VIP Refresher: Calculus

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Integral calculus

❒ **Primitive function** – The primitive function of a function f, noted F and also known as an antiderivative, is a differentiable function such that:

$$\boxed{F' = f}$$

❒ **Integral** – Given a function f and an interval $[a,b]$, the integral of f over $[a,b]$, noted $\int_a^b f(x)dx$, is the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$, and can be computed with the primitive of f as follows:

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

❒ **Integration by parts** – Given two functions f, g on the interval $[a,b]$, we can integrate by parts the quantity $\int_a^b f(x)g'(x)dx$ as follows:

$$\boxed{\int_a^b f(x)g'(x)dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx}$$

❒ **Rational primitive functions** – The table below sums up the main rational functions associated to their primitives. We will omit the additive constant C associated to all those primitives.

Function f	**Primitive** F
a	ax
x^a	$\dfrac{x^{a+1}}{a+1}$
$\dfrac{1}{x}$	$\ln\|x\|$
$\dfrac{1}{1+x^2}$	$\arctan(x)$
$\dfrac{1}{1-x^2}$	$\dfrac{1}{2}\ln\left\|\dfrac{x+1}{x-1}\right\|$

❒ **Irrational primitive functions** – The table below sums up the main rational functions associated to their primitives. We will omit the additive constant C associated to all those primitives:

Function f	**Primitive** F
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$-\dfrac{1}{\sqrt{1-x^2}}$	$\arccos(x)$
$\dfrac{x}{\sqrt{x^2-1}}$	$\sqrt{x^2-1}$

❒ **Exponential primitive functions** – The table below sums up the main exponential functions associated to their primitives. We will omit the additive constant C associated to all those primitives.

Function f	**Primitive** F
$\ln(x)$	$x\ln(x) - x$
$\exp(x)$	$\exp(x)$

❒ **Trigonometric primitive functions** – The table below sums up the main trigonometric functions associated to their primitives. We will omit the additive constant C associated to all those primitives.

Function f	**Primitive** F
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\tan(x)$	$-\ln\|\cos(x)\|$
$\dfrac{1}{\cos(x)}$	$\ln\left\|\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\right\|$
$\dfrac{1}{\sin(x)}$	$\ln\left\|\tan\left(\dfrac{x}{2}\right)\right\|$
$\dfrac{1}{\tan(x)}$	$\ln\|\sin(x)\|$

Laplace transforms

❒ **Definition** – The Laplace transform of a given function f defined for all $t \geqslant 0$ is noted $\mathscr{L}(f)$, and is defined as:

$$\boxed{\mathscr{L}(f) = F(s) = \int_0^{+\infty} e^{-st} f(t)dt}$$

Remark: we note that $f(t) = \mathscr{L}^{-1}(F)$ where $\mathscr{L}^{-1}$ is the inverse Laplace transform.

❒ **Main properties** – The table below sums up the main properties of the Laplace transform:

	Property	t-domain	s-domain
	Linearity	$\alpha f(t)+\beta g(t)$	$\alpha F(s)+\beta G(s)$
t-domain	Integral	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
t-domain	First derivative	$f'(t)$	$sF(s)-f(0)$
t-domain	Second derivative	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
t-domain	n^{th} derivative	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-...-sf^{(n-2)}(0)-f^{(n-1)}(0)$
s-domain	Integral	$\frac{f(t)}{t}$	$\int_s^{+\infty}F(\sigma)d\sigma$
s-domain	First derivative	$tf(t)$	$-F'(s)$
s-domain	Second derivative	$t^2f(t)$	$F''(s)$
s-domain	n^{th} derivative	$t^nf(t)$	$(-1)^nF^{(n)}(s)$

❒ **Common transform pairs** – The table below sums up the most common Laplace transform pairs:

t-domain	s-domain
a	$\frac{a}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$

❒ **Main operations** – The table below sums up the main operations of the Laplace transform:

Operation	t-domain	s-domain
Unit step function	$u(t-a)$	$\frac{e^{-as}}{s}$
Dirac delta function	$\delta(t-a)$	e^{-as}
s-shift	$e^{at}f(t)$	$F(s-a)$
t-shift	$u(t-a)f(t-a)$	$e^{-as}F(s)$